Structural Equation Models: Some Selected Examples

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Structural Equation Models: Some Selected Examples

Introduction

- In this module and several that follow, we examine a selection of classic examples of structural equation models with continuous variables.
- These models illustrate a number of core techniques and problem issues in structural equation modeling.

The Lawley-Maxwell Confirmatory Factor Analysis

- In the early days of structural equation modeling, LISREL was the only available program for doing structural equation modeling.
- A number of the examples in the LISREL manual used data from publications which included only information on the correlation matrix.
- Moreover, in the case of factor analysis, the typical procedure was to factor analyze the correlation matrix rather than the covariance matrix.
- The problem was that the distribution theory LISREL was using assumed that the data matrix being analyzed was a covariance matrix.

The Lawley-Maxwell Confirmatory Factor Analysis

- Recall that the ML discrepancy function is

$$F_{ML}(\mathbf{S}, \mathbf{M}(\theta)) = \log|\mathbf{M}(\theta)| - \log|\mathbf{S}| + \mathrm{Tr}(\mathbf{S}\mathbf{M}(\theta)^{-1}) - p \tag{1}$$

  witn p is the number of manifest variables.
- This function tests the fit of a model $\mathbf{M}$ to a population covariance matrix $\boldsymbol{\Sigma}$, based on the multivariate distribution of the elements of $\mathbf{S}$.
- If a correlation matrix is all that is available, what can you do?
- Early users of LISREL did what the LISREL manual did — simply input the correlation matrix and tell the program it was a covariance matrix.

The Lawley-Maxwell Confirmatory Factor Analysis

- Consider the sample correlation matrix **R** computed from a sample covariance matrix **S**.
- We can get **R** from **S**, but we can't get **S** from **R** unless we have standard deviations available.
- This means, pretty obviously, that there is more information in **S** than in **R**, and so although **R** looks like a covariance matrix (because it is!), its non-redundant elements are different in number, and have a different multivariate distribution, than those of **S**.

The Lawley-Maxwell Confirmatory Factor Analysis

- Consider the simple case of a 2×2 sample covariance matrix $\mathbf{S}$.
- It has 3 non-redundant elements in it.
- Now suppose I were to transform S into its corresponding correlation matrix $\mathbf{R}$.
- How many non-redundant elements would $\mathbf{R}$ have in it? Just one.
- Clearly, the tri-variate distribution of the non-redundant elements of $\mathbf{S}$ in this case cannot be the same as the univariate distribution of the single random element of $\mathbf{R}$.

The Lawley-Maxwell Confirmatory Factor Analysis

- The LISREL manual detailed several examples in which a correlation matrix was incorrectly analyzed as a covariance matrix.
- It was easy to get LISREL (or any other program) to do this. Simply input the correlation matrix and tell the program it is a covariance matrix!
- There were occasional statements that one shouldn't analyze correlations, but these were never rendered in a clear style that would lead one to conclude that LISREL output was incorrect.
- Many LISREL users came away with the mistaken notion that correlations should never be analyzed on statistical grounds, rather than the more reasonable conclusion that correlations couldn't be analyzed correctly by LISREL.

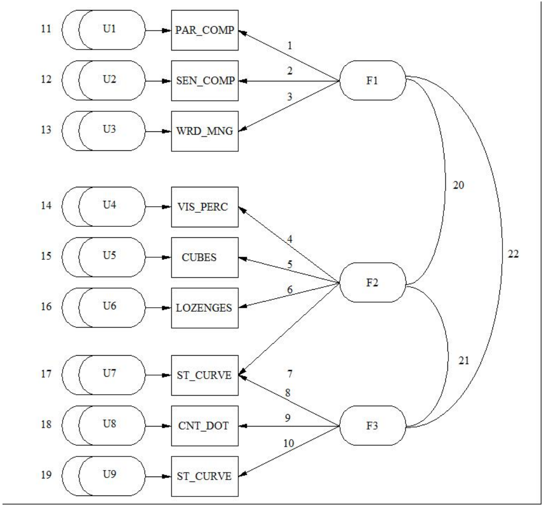
The Lawley-Maxwell Confirmatory Factor Analysis

- By the early 1990's, several computer programs *could* correctly analyze correlations, but LISREL still could not.
- What was particularly intriguing about the situation was that the classic textbook *Factor Analysis as a Statistical Method*, by Lawley and Maxwell (1971), actually devoted a chapter explaining why a confirmatory factor analysis that treated a correlation matrix as if it were a covariance matrix would often generate correct parameter estimates, but wrong standard errors.
- The Lawley-Maxwell chapter contained a fully worked numerical example.

The Lawley-Maxwell Confirmatory Factor Analysis

- Lawley and Maxwell(1971) gave formulae for computing standard errors when a covariance matrix is analyzed, and provided an alternative method for computing standard errors when the sample correlation matrix is analyzed.
- The formulae are illustrated with a numerical example, the results of which are presented in their Tables 7.9 (page 99) and 7.10 (page 102).
- The Lawley-Maxwell example was based on an analysis in a paper by Jöreskog and Lawley (1968), which used data from Holzinger and Swineford (1939). Nine psychological tests were administered to 72 seventh and eighth grade students.
- The path diagram is on the next slide.
- Lawley and Maxwell did not use ULI (Unit Loading Identification) constraints in their model. Rather, in keeping with standard practice in factor analysis, they parameterized their model to have unit variances.

The Lawley-Maxwell Confirmatory Factor Analysis



The Lawley-Maxwell Confirmatory Factor Analysis

- On the next few slides, I give the factor loadings and standard errors (incorrect version in parentheses) tables from Lawley, followed by output from my own program SEPATH that correctly analyzes correlation matrices directly.
- As you can see (keeping in mind that variables are presented in different order by Lawley), the results agree with his.
- SEPATH has the ability to override the correct analysis and analyze the correlation matrix incorrectly as if it were a covariance matrix, so we can also reproduce the incorrect version of standard errors given by Lawley.

The Lawley-Maxwell Confirmatory Factor Analysis

Table 7.9 RESTRICTED MAXIMUM LIKELIHOOD SOLUTION FOR CONFIRMATION SAMPLE

Variate	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\Psi}$
1	0·68	0	0	0·54
2	0·34	0	0	0·88
3	0·66	0	0	0·57
4	0	0·91	0	0·18
5	0	0·87	0	0·25
6	0	0·82	0	0·32
7	0	0	0·65	0·58
8	0	0	0·92	0·15
9	0·67	0	0·19	0·39

Factor correlation matrix $\hat{\Phi}$

1·00	0·55	0·47
	1·00	0·09
		1·00

The Lawley-Maxwell Confirmatory Factor Analysis

Table 7.10 STANDARD ERRORS FOR ESTIMATES OF *Table 7.9*

Variate	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\Psi}$
1	0·09(0·12)	0	0	0·12(0·12)
2	0·12(0·13)	0	0	0·08(0·15)
3	0·09(0·12)	0	0	0·12(0·12)
4	0	0·04(0·10)	0	0·06(0·06)
5	0	0·04(0·10)	0	0·07(0·06)
6	0	0·05(0·10)	0	0·08(0·07)
7	0	0	0·10(0·13)	0·13(0·14)
8	0	0	0·11(0·14)	0·21(0·20)
9	0·11(0·13)	0	0·13(0·13)	0·11(0·10)

The Lawley-Maxwell Confirmatory Factor Analysis

	Model Estimates (Lawley)			
	Parameter Estimate	**Standard Error**	T Statistic	Prob. Level
(VERBAL)-1->[PAR_COMP]	0.908	0.036	25.519	0.000
-2->[SEN_COMP]	0.867	0.041	21.414	0.000
-3->[WRD_MNG]	0.824	0.047	17.656	0.000
(VISUAL)-4->[VIS_PERC]	0.679	0.086	7.869	0.000
-5->[CUBES]	0.341	0.121	2.827	0.005
-6->[LOZENGES]	0.659	0.089	7.441	0.000
-7->[ST_CURVE]	0.670	0.113	5.956	0.000
(SPEED) -8->[ST_CURVE]	0.192	0.129	1.483	0.138
-9->[CNT_DOT]	0.924	0.111	8.296	0.000
-10->[ADDITION]	0.651	0.103	6.291	0.000
(U1)-->[PAR_COMP]				
(U2)-->[SEN_COMP]				
(U3)-->[WRD_MNG]				
(U4)-->[VIS_PERC]				
(U5)-->[CUBES]				
(U6)-->[LOZENGES]				
(U7)-->[ST_CURVE]				
(U8)-->[CNT_DOT]				
(U9)-->[ADDITION]				
(U1)-11-(U1)	0.175	0.065	2.715	0.007
(U2)-12-(U2)	0.248	0.070	3.525	0.000
(U3)-13-(U3)	0.321	0.077	4.170	0.000
(U4)-14-(U4)	0.538	0.117	4.587	0.000
(U5)-15-(U5)	0.884	0.082	10.740	0.000
(U6)-16-(U6)	0.566	0.117	4.855	0.000
(U7)-17-(U7)	0.392	0.107	3.655	0.000
(U8)-18-(U8)	0.146	0.206	0.708	0.479
(U9)-19-(U9)	0.577	0.135	4.285	0.000
(VERBAL)-20-(VISUAL)	0.552	0.111	4.974	0.000
(VISUAL)-21-(SPEED)	0.474	0.143	3.324	0.001
(VERBAL)-22-(SPEED)	0.088	0.133	0.661	0.509

The Lawley-Maxwell Confirmatory Factor Analysis

	Model Estimates (Lawley)			
	Parameter Estimate	Standard Error	T Statistic	Prob. Level
(VERBAL)-1->[PAR_COMP]	0.908	0.095	9.513	0.000
-2->[SEN_COMP]	0.867	0.098	8.870	0.000
-3->[WRD_MNG]	0.824	0.100	8.229	0.000
(VISUAL)-4->[VIS_PERC]	0.680	0.119	5.701	0.000
-5->[CUBES]	0.341	0.130	2.630	0.009
-6->[LOZENGES]	0.659	0.120	5.499	0.000
-7->[ST_CURVE]	0.670	0.135	4.978	0.000
(SPEED) -8->[ST_CURVE]	0.192	0.130	1.471	0.141
-9->[CNT_DOT]	0.924	0.142	6.506	0.000
-10->[ADDITION]	0.651	0.131	4.974	0.000
(U1)-->[PAR_COMP]				
(U2)-->[SEN_COMP]				
(U3)-->[WRD_MNG]				
(U4)-->[VIS_PERC]				
(U5)-->[CUBES]				
(U6)-->[LOZENGES]				
(U7)-->[ST_CURVE]				
(U8)-->[CNT_DOT]				
(U9)-->[ADDITION]				
(U1)-11-(U1)	0.175	0.060	2.919	0.004
(U2)-12-(U2)	0.248	0.064	3.884	0.000
(U3)-13-(U3)	0.321	0.070	4.592	0.000
(U4)-14-(U4)	0.538	0.120	4.486	0.000
(U5)-15-(U5)	0.884	0.154	5.746	0.000
(U6)-16-(U6)	0.566	0.122	4.655	0.000
(U7)-17-(U7)	0.392	0.103	3.813	0.000
(U8)-18-(U8)	0.146	0.205	0.712	0.477
(U9)-19-(U9)	0.577	0.140	4.125	0.000
(VERBAL)-20-(VISUAL)	0.552	0.111	4.974	0.000
(VISUAL)-21-(SPEED)	0.474	0.143	3.324	0.001
(VERBAL)-22-(SPEED)	0.088	0.133	0.661	0.509

The Lawley-Maxwell Confirmatory Factor Analysis

- As we have already pointed out, there is an alternative parameterization of the Lawley-Maxwell confirmatory analysis, in which the factor variances are estimated and a ULI (Unit Loading Identification) constraint is employed.
- It should be emphasized that the parameter estimates for this model will generally be different from those in the standardized factor model, and the standard errors will also be different.
- Such a model can be analyzed correctly with a correlation matrix, and in general the standard errors obtained with this parameterization will (like the parameter estimates themselves) be different *both from the alternative parameterization and from the incorrect processing of a correlation matrix as if it were a covariance matrix*.

The Lawley-Maxwell Confirmatory Factor Analysis

- In general, if a correlation matrix is processed incorrectly as if it were a covariance matrix in a *scale-invariant* factor model, the parameter estimates will be correct. However, the standard errors will not be correct.
- In other words, if we contemplate the Lawley-Maxwell model, and restrict ourselves to just the 4 loadings on the *Visual* factor, there are (at least) 4 possible versions of these loadings when a correlation matrix is all that is available:
  1. *Unit Variance Correct (UVC).* The factor has unit variance, and the standard errors for 4 loadings are correctly estimated.
  2. *Unit Variance Incorrect (UVI).* The factor has unit variance, but the correlation matrix is treated incorrectly as a covariance matrix, and the standard errors are incorrectly estimated.
  3. *ULI Correct (ULIC).* The factor variance is a free parameter. Instead, a Unit Loading Identification (ULI) constraint is used on the first variable. The 3 other loadings have standard errors that are correctly estimated.
  4. *ULI Incorrect (ULII).* The factor variance is a free parameter. Instead, a Unit Loading Identification (ULI) constraint is used on the first variable. The 3 other loadings have standard errors that are incorrectly estimated by treating the correlation matrix as a covariance matrix.
- SEPATH can generate all 4 kinds of estimates, and here they are, together with the standard errors.

The Lawley-Maxwell Confirmatory Factor Analysis

Loading	***Condition***			
	UVC	*UVI*	*ULIC*	*ULII*
VIS_PERC	0.680 (0.086)	0.680 (0.119)	1	1
CUBES	0.341 (0.121)	0.341 (0.130)	0.502 (0.185)	0.502 (0.202)
LOZENGES	0.659 (0.089)	0.659 (0.120)	0.970 (0.172)	0.969 (0.219)
ST_CURVE	0.670 (0.113)	0.670 (0.135)	0.987 (0.217)	0.987 (0.250)

The Lawley-Maxwell Confirmatory Factor Analysis

- Let's analyze these data with Mplus.
- The default parameterization in Mplus uses ULI constraints.
- In the input below, we set sample size at 71, so that Mplus standard errors, computed with a denominator of n, will match those of other programs that use $n - 1$.

The Lawley-Maxwell Confirmatory Factor Analysis

```
TITLE: LAWLEY-MAXWELL CONFIRMATORY MODEL
DATA: FILE is lawley.dat;
      TYPE IS CORRELATION;
      NOBSERVATIONS=71;
VARIABLE: NAMES ARE VIS_PERC CUBES LOZENGES PAR_COMP
          SEN_COMP WRD_MNG ADDITION CNT_DOT ST_CURVE;
MODEL: VERBAL BY PAR_COMP SEN_COMP WRD_MNG;
       VISUAL BY VIS_PERC CUBES LOZENGES ST_CURVE;
       SPEED BY ST_CURVE CNT_DOT ADDITION;
OUTPUT: STANDARDIZED;
```

The Lawley-Maxwell Confirmatory Factor Analysis

The basic loading output, shown below, has the incorrect standard errors.

```
MODEL RESULTS

                                                Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 VERBAL   BY
    PAR_COMP           1.000      0.000    999.000    999.000
    SEN_COMP           0.955      0.099      9.638      0.000
    WRD_MNG            0.908      0.101      8.950      0.000

 VISUAL   BY
    VIS_PERC           1.000      0.000    999.000    999.000
    CUBES              0.502      0.202      2.483      0.013
    LOZENGES           0.970      0.220      4.416      0.000
    ST_CURVE           0.987      0.250      3.947      0.000

 SPEED    BY
    ST_CURVE           1.000      0.000    999.000    999.000
    CNT_DOT            4.825      3.423      1.410      0.159
    ADDITION           3.396      2.307      1.472      0.141
```

The Lawley-Maxwell Confirmatory Factor Analysis

However, the standardized output (STDXY Standardization), transformed to unit variance factors, matches the correct loading obtained with a unit variance factor parameterization in SEPATH.

```
STANDARDIZED MODEL RESULTS


STDYX Standardization

                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 VERBAL   BY
    PAR_COMP           0.908      0.036     25.520      0.000
    SEN_COMP           0.867      0.041     21.413      0.000
    WRD_MNG            0.824      0.047     17.656      0.000

 VISUAL   BY
    VIS_PERC           0.679      0.086      7.868      0.000
    CUBES              0.341      0.121      2.827      0.005
    LOZENGES           0.659      0.089      7.442      0.000
    ST_CURVE           0.670      0.113      5.957      0.000

 SPEED    BY
    ST_CURVE           0.192      0.129      1.483      0.138
    CNT_DOT            0.924      0.111      8.296      0.000
    ADDITION           0.651      0.103      6.290      0.000
```

The Lawley-Maxwell Confirmatory Factor Analysis

- Mplus evidently obtains its standardized solution by taking the unstandardized solution obtained with the ULI parameterization, transforming it to a standardized solution, and then revising the standard errors.
- Let's see what happens when we directly program the standardized solution by demanding unit variances for our factors.

The Lawley-Maxwell Confirmatory Factor Analysis

```
TITLE: LAWLEY-MAXWELL CONFIRMATORY MODEL
DATA: FILE is lawley.dat;
      TYPE IS CORRELATION;
      NOBSERVATIONS=71;
VARIABLE: NAMES ARE VIS_PERC CUBES LOZENGES PAR_COMP
          SEN_COMP WRD_MNG ADDITION CNT_DOT ST_CURVE;
MODEL: VERBAL BY PAR_COMP*;
       VERBAL BY SEN_COMP*;
       VERBAL BY WRD_MNG*;
       VISUAL BY VIS_PERC*;
       VISUAL BY CUBES*;
       VISUAL BY LOZENGES*;
       VISUAL BY ST_CURVE*;
       SPEED BY ST_CURVE*;
       SPEED BY CNT_DOT*;
       SPEED BY ADDITION*;
       VISUAL@1;
       VISUAL WITH VERBAL*;
       VISUAL WITH SPEED*;
       VERBAL@1;
       VERBAL WITH SPEED*;
       SPEED@1;
OUTPUT: STANDARDIZED;
```

The Lawley-Maxwell Confirmatory Factor Analysis

```
MODEL RESULTS

                                                  Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 VERBAL   BY
    PAR_COMP           0.902      0.095      9.513      0.000
    SEN_COMP           0.861      0.097      8.870      0.000
    WRD_MNG            0.818      0.099      8.229      0.000

 VISUAL   BY
    VIS_PERC           0.675      0.118      5.700      0.000
    CUBES              0.339      0.129      2.630      0.009
    LOZENGES           0.654      0.119      5.499      0.000
    ST_CURVE           0.666      0.134      4.979      0.000

 SPEED    BY
    ST_CURVE           0.190      0.129      1.471      0.141
    CNT_DOT            0.918      0.141      6.506      0.000
    ADDITION           0.646      0.130      4.973      0.000


 VISUAL   WITH
    VERBAL             0.552      0.111      4.973      0.000
    SPEED              0.474      0.143      3.324      0.001

 VERBAL   WITH
    SPEED              0.088      0.133      0.661      0.509

 Variances
    VERBAL             1.000      0.000    999.000    999.000
    VISUAL             1.000      0.000    999.000    999.000
    SPEED              1.000      0.000    999.000    999.000
```

The Lawley-Maxwell Confirmatory Factor Analysis

- Although the solution clearly is standardized, and shows the factor variances fixed at one, the estimates are different from those obtained by other software *and by Mplus in its standardized solution*.
- There is a reason for this, but it is sufficiently obscure that very few users would be capable of understanding its genesis.
- The standard errors are incorrect. They appear to be obtained by analyzing the correlation matrix as if it were a covariance matrix.
- However, the standardized solution printed after the direct unit variance model has both the correct loadings and standard errors for a proper estimation from a correlation matrix!

The Lawley-Maxwell Confirmatory Factor Analysis

- To summarize:
  1. Although the DATA section of the Mplus input clearly labels the correlation matrix to be a correlation matrix, the base Mplus output processes it as if it were a covariance matrix. Incorrect standard errors are printed with both the unit variance and ULI parameterizations.
  2. The MODEL RESULTS loadings are incorrect when the unit variance paramaterization is selected to produce a standardized factor analysis solution directly. They disagree with those printed by Mplus with the standardized solution, and with the results from other programs.
  3. When a standardized solution is requested, *and the base parameterization is employed*, Mplus returns correct loadings and standard errors with unit variance factors.

Model Invariance Properties

- In his classic paper on the proper processing of correlation matrices, Cudeck (1989) discussed *model invariance* properties and how they related to what would happen when a correlation matrix was treated improperly as a covariance matrix.
- Subsequently, Browne and Shapiro (1991) presented methods that can be used to detect whether a model fails to meet certain invariance properties.
- Understanding model invariance can help explain some (but not all) of the phenomena we just witnessed in connection with the Lawley example, and I've assigned the Cudeck paper for reading.
- Keep in mind, however, that Cudeck uses the somewhat confusing notation $\mathbf{\Sigma} = \mathbf{\Sigma}(\boldsymbol{\theta})$ to stand for a covariance structure model, rather than $\mathbf{\Sigma} = \mathbf{M}(\boldsymbol{\theta})$ which many more recent authors prefer.

Model Invariance Properties

- We begin with some definitions.
- A covariance structure for a $p \times p$ covariance matrix $\mathbf{\Sigma}$ is a symmetric, matrix valued function $\mathbf{M}(\gamma)$ of a $q \times 1$ parameter vector γ.
- The null hypothesis in covariance structure modeling is thus

$$\mathbf{\Sigma} = \mathbf{M}(\gamma) \tag{2}$$

Model Invariance Properties

- A correlation structure is a symmetric, matrix-valued function $\mathbf{R}(\boldsymbol{\theta})$ of a $q \times 1$ parameter vector $\boldsymbol{\theta}$ with the restriction that $diag(\mathbf{R}(\boldsymbol{\theta})) = diag(\mathbf{I})$. The null hypothesis for a correlation structure model is that, for a population correlation matrix $\mathbf{P}$,

$$\mathbf{P} = \mathbf{R}(\boldsymbol{\theta}) \tag{3}$$

Model Invariance Properties

- A covariance structure model $\mathbf{M}(\gamma)$ is scale-invariant or invariant under changes of scale if for any diagonal matrix $\mathbf{D}_\alpha = \{\alpha_j\}$ with no zero entries, there exists another parameter vector γ^*, such that
$$\mathbf{M}(\gamma^*) = \mathbf{D}_\alpha \mathbf{M}(\gamma) \mathbf{D}_\alpha \tag{4}$$
- In other words, scale invariance implies that if a model fits a covariance matrix, it will fit any rescaling of that covariance matrix for some parameter vector.

Model Invariance Properties

- A weaker form of invariance is invariance under a constant scaling factor.
- Most covariance structures of any practical interest are at least invariant under a constant scaling factor.
- Some of the non-centrality based fit indices developed by Steiger (1989, 1990) require that the model fitted be invariant under a constant scaling factor (ICSF).
- A covariance structure model $\mathbf{M}(\gamma)$ is invariant under a constant scaling factor (ICSF) if, for any positive constant τ, there exists another parameter vector γ^*, such that $\mathbf{M}(\gamma^*) = \tau\mathbf{M}(\gamma)$.

Model Invariance Properties

- Consider a covariance structure model $\boldsymbol{\Sigma} = \mathbf{M}(\boldsymbol{\gamma})$. A parameter $\gamma_i \in \boldsymbol{\gamma}$ is scale-free if
  1. The model is scale-invariant, and
  2. for choices of $\mathbf{D}_\alpha$ in Equation 4, $\gamma_i^* = \gamma_i$.
- Cudeck (1989, p. 318–319) gives examples of
  1. A confirmatory factor model that is scale-invariant,
  2. A scale-free parameter in such a model, and
  3. A confirmatory factor model that is not scale-invariant.

Model Invariance Properties

- Suppose a covariance structure factor model $\mathbf{M}(\gamma) = \mathbf{\Lambda\Phi\Lambda}' + \mathbf{\Psi}$ is scale-invariant.
- Then the model can be re-expressed as

$$\begin{aligned} \mathbf{M}(\gamma) &= \mathbf{D}_\sigma \mathbf{R}(\theta) \mathbf{D}_\sigma && (5) \\ &= \mathbf{D}_\sigma (\mathbf{\Lambda}^* \mathbf{\Phi}^* \mathbf{\Lambda}^{*\prime} + \mathbf{\Psi}^*) \mathbf{D}_\sigma && (6) \end{aligned}$$

  where $\mathbf{R}(\theta)$ is a standardized common factor model, and the elements of $\mathbf{D}_\sigma$ are population standard deviations.
- If this model $\mathbf{R}(\theta)$ fits the correlation matrix, then if the model is scale-invariant, one must be able to find $\mathbf{D}_\sigma$ such that $\mathbf{M}(\gamma)$ fits the population covariance matrix.
- If one's main interest is testing a correlation structure model, then the values in $\mathbf{D}_\alpha$ are "nuisance parameters". If the correlation structure fits, the covariance structure can automatically be made to fit by setting $\mathbf{D}_\alpha$ to the value required to match the diagonal of the input covariance matrix.
- The key aspect of this latter fact is that this will be true even if the input matrix is actually a correlation matrix. The fact that the estimates in $\mathbf{D}_\alpha$ will not actually be estimates of the population standard deviations will be unimportant.

Model Invariance Properties

- Consequently, correlation structure models can be fit as covariance structure models of the form
$$\mathbf{M}(\gamma) = \mathbf{D}_\sigma \mathbf{R}(\theta) \mathbf{D}_\sigma \tag{7}$$
with
$$\boldsymbol{\gamma}' = \left[\begin{array}{c|c} \boldsymbol{\theta}' & \boldsymbol{\sigma}' \end{array} \right] \tag{8}$$
and $\boldsymbol{\sigma}$ containing the diagonal elements of $\mathbf{D}_\sigma$.
- The vector $\boldsymbol{\theta}$ contains the parameters of the correlation structure, while $\boldsymbol{\sigma}$ contains the nuisance parameters.

Model Invariance Properties

- Cudeck pointed out that if a covariance structure model is scale-invariant, the chi-square test statistic will be the same regardless of whether the model is fit to the actual covariance matrix or the correlation matrix.
- If the model is not scale-invariant, then different scales will result in different fit values. Consequently, the advice proffered in numerous books and computer manuals about casually rescaling a variable in order to improve iteration should be accompanied by a caveat. If the model is not scale-invariant, then the fit indices and statistical test results will generally differ after rescaling.
- The standard errors will generally be incorrect if methods based on an analysis of a covariance matrix are applied incorrectly directly to a correlation matrix.

Testing for Scale Invariance

- Browne and Shapiro (1991) defined a reflector matrix corresponding to each of the typical discrepancy functions for continuous variables.
- They then showed how this reflector matrix could be analyzed to shed light on the invariance properties of a particular model when fitted with a particular discrepancy function.
- We will begin by defining the reflector matrix corresponding to each type of discrepancy function.

Testing for Scale Invariance

- Let $\mathbf{S}$ stand for the sample covariance matrix and $\hat{\Sigma}$ stand for the reproduced matrix, i.e., $\mathbf{M}(\gamma)$ at the minimum point for that discrepancy function.
- Furthermore, define $\hat{\mathbf{E}} = \mathbf{S} - \hat{\Sigma}$. Then the following reflector matrices (among others) can be defined.

$$\begin{aligned} \hat{\Omega}_{ML} &= -\hat{\Sigma}^{-1}\hat{\mathbf{E}} && (9) \\ \hat{\Omega}_{LS} &= -\hat{\mathbf{E}}\hat{\Sigma} && (10) \\ \hat{\Omega}_{GLS} &= (\mathbf{S}^{-1}\hat{\Sigma} - \mathbf{I})\mathbf{S}^{-1}\hat{\Sigma} && (11) \end{aligned}$$

- In the next slide, we examine how the reflector matrices can be used to test invariance properties.

Testing for Scale Invariance

- Browne and Shapiro (1991) give several corollaries which establish properties of reflector matrices implied by various types of scale invariances. Their results include the following corollaries:
  1. Let $\mathbf{M}(\gamma)$ be invariant under a constant scaling factor. Then the sum of the diagonal elements of $\hat{\mathbf{\Omega}}$ is zero.
  2. Let $\mathbf{M}(\gamma)$ be scale-invariant. Then all diagonal elements of $\hat{\mathbf{\Omega}}$ are zero.
- We can assess invariance under a constant scaling factor by examining the trace of the reflector matrix, and we can assess scale invariance by examining the maximum diagonal element.
- Of course, in some cases lack of invariance is obvious from the model itself. If the model-fitting software does not provide the reflector matrices or related indices, one may simply try rescaling the input matrix by a constant or by a set of scale factors.

Constraint Interaction

- In an earlier lecture, I alluded briefly to the options one has in standard SEM software to parameterize the factors in a factor model as having either unit variance, or a fixed loading on one variable (but not both).
- One can also parameterize the "unique factors" as having unit variance, and have their loading on their observed variable be a free parameter, or have the loading be 1 and estimate the unique variance.
- When the factor model is analyzed in isolation, either option is available.
- As I mentioned before, there are good *practical and theoretical* reasons for fixing the variance of exogenous latent variables (be they common or unique factors) at unity:
  1. On the practical side, it is no longer possible to iterate to a solution that estimates negative factor variances, and,
  2. On the theoretical side, if (and it is a very big "if") the software can correctly analyze correlations (or, equivalently, correctly standardize the manifest variables), then a fully standardized factor solution can be obtained, comparable with what the originators of factor analysis did.

Constraint Interaction

- The Unit Loading Identification (ULI) constraints are intended to be employed solely to establish identification.
- The standard dictum is that each common factor in the factor model should have one ULI constraint to establish identification of that factor's variance.
- We have already seen that, even with standard software, this restriction is generally not necessary if the confirmatory common factor model is in isolation, or if the common factors are exogenous. ULI constraints can be replaced by fixed unit variances for the common factors in such cases.

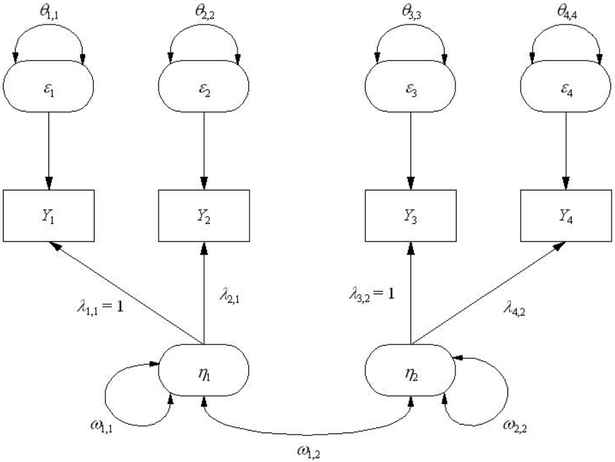
Constraint Interaction

- Things are much more problematic if the CFA model is embedded in a structural equation model in which its factors are endogenous.
- Standard SEM software does not input variance parameters for endogenous variables. Rather, the variances of endogenous variables are established as a consequence of the variances and covariances of the exogenous variables in the model,m and the structural paths leading from them to the endogenous variables (possibly indirectly through other endogenous variables).

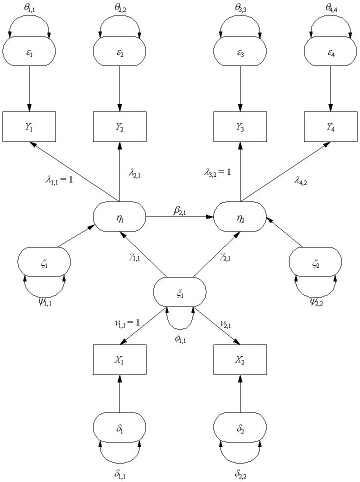
ULI Constraints and Identification

- Let's look at a simple CFA model, analyze its behavior in isolation, and then embed it in a larger, very familiar structural equation model.

ULI Constraints and Identification



ULI Constraints and Identification



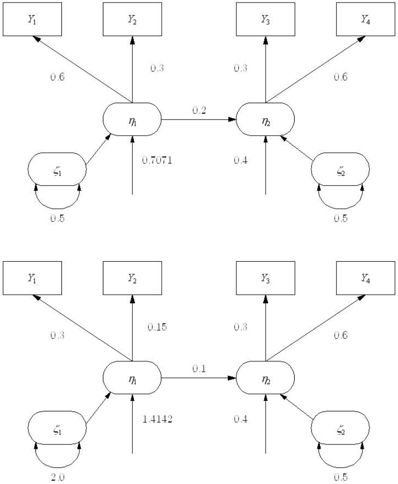
ULI Constraints and Identification

- Examining these models, we note first that latent variable η_1 is never observed, and so its variability may only be inferred from two sources:
  1. the variances and covariances of the variables with paths leading to η_1,
  2. the values of the path coefficients leading to η_1.

ULI Constraints and Identification

- The variances of η_1 and η_2 are not uniquely defined, and are free to vary unless some constraints are imposed on the free parameters in our confirmatory model.
- To see why, suppose that the ULI constraints *were removed* from $\lambda_{1,1}$ and $\lambda_{2,1}$, and that, by some combination of circumstances, the paths leading to η_1 and η_2 had values that caused η_1 to have a variance of 1.
- Suppose further that under these circumstances, the values .6, .3, .6, and .3 for parameters $\lambda_{1,1}$,$\lambda_{2,1}$,$\lambda_{3,2}$, and $\lambda_{4,2}$ lead to an optimal fit of the model to the data.
- The upper diagram in the figure on the next slide shows this situation.

ULI Constraints and Identification



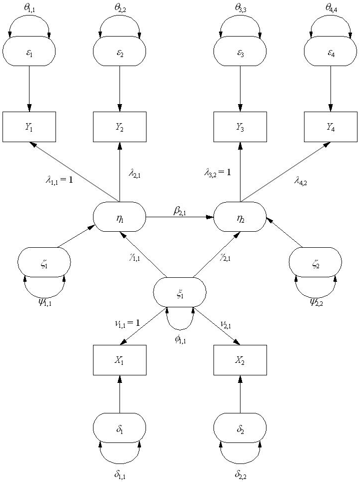
ULI Constraints and Identification

- Next, imagine we wished the variance of η_1 to be some value other than 1, say, 4.
- Quadrupling a variable's variance can be accomplished by doubling its standard deviation, or doubling every value of the variable.
- To achieve this, while maintaining the identical numbers arriving at Y_1, Y_2 and η_2 from η_1, we need only double all values on paths leading to η_1, while simultaneously halving all values ($\lambda_{1,1}$, $\lambda_{2,1}$, and $\beta_{2,1}$) on paths leading away from η_1.

ULI Constraints and Identification

- Every number emerging from η_1 is doubled, but is "passed through" coefficients that are now exactly half what they were. So the numbers emerging at Y_1, Y_2, and η_2 are the same as they were.
- Because $\psi_{1,1}$ and $\gamma_{1,1}$ are free parameters that are attached to unidirectional paths, we can alter them (to halve the values of all numbers arriving at η_1) without affecting anything in the paths leading into our measurement model when it is embedded in a larger model, as shown on the next slide.

ULI Constraints and Identification



ULI Constraints and Identification

- The pipeline metaphor has helped us to see why ULI constraints are necessary to identify the coefficients in a path model. Without the application of such a constraint, we can see that the variances of our latent variables could be any positive value.
- Now, the choice of a constraint to apply is, in a sense, arbitrary. Why do you think the "powers that be" chose to set one loading to 1, rather than to, say, 2? Is there some *other*, more natural constraint that people might have chosen?

ULI Constraints and Identification

- How about constraining the endogenous latent variables to have variances of 1?

ULI Constraints and Identification

- When we change the particular constraint we employ to achieve identification, what things (if any) remain constant?

ULI Constraints and Identification

Characteristics of Properly Deployed ULI Constraints

- By using the pipeline metaphor, we've deduced some things.
  1. When a ULI constraint is applied to a parameter, the primary goal is simply to establish identification, and the precise value that the parameter is fixed to will not affect the fit of the model. Specifically, one could use the value 2.0 instead of 1.0, and the test statistic for the model would remain the same, because the fit of the model is *invariant under change of scale of its latent variables.*
  2. The particular manifest variable chosen for the ULI constraint for any latent variable should not affect model fit. In the present example, fit will be the same if we constrain either $\lambda_{1,1}$ or $\lambda_{2,1}$ (but not both).
  3. Path coefficients leading from a latent variable have the same relative magnitude regardless of the fixed value used in a ULI. Their absolute magnitude will go up or down depending on the fixed value used in the ULI. So, for example, if one changes the 1.0 to a fixed value of 2.0, all path coefficients leading from the latent variable will double.
  4. Any multiplicative change in the ULI constraint applied to a path coefficient will be mirrored by a corresponding division of the standard deviation of the latent variable the path leads from, and a corresponding division of path coefficients leading to the latent variable.

ULI Constraints and Identification

Characteristics of Properly Deployed ULI Constraints

- The above properties reflect the way ULI constraints are supposed to work in practice. *The constraints are intended to be essentially arbitrary values imposed solely to achieve identification, and are not intended to have any substantive impact on model fit or model interpretation.*
- There seems to be some confusion in the literature about the latter point.
- Some sources make a statement to the effect that a ULI constraint for the loading of a particular manifest variable fixes the scale of the latent variable to be "the same as the manifest variable."

ULI Constraints and Identification

Characteristics of Properly Deployed ULI Constraints

- This misconception has led to the use of the term *reference variable* to refer to the manifest variable with the ULI attached.
- This view not quite right — if a value of unity is employed, the variance of the latent variable is fixed to the variance of the *common part* of the manifest variable which has the ULI constraint.
- Moreover, as we have already seen, all other loadings emanating from the latent variable move up or down in concert with the value selected for the ULI constraint, and the variance of the common part is itself determined by the choice of variables in the measurement model.
- The key issue here is that residual variance includes error variance and unique variance, so fixing the metric of the latent variable to an observed variable's common variance has dubious value.

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- With these goals in mind, it seems reasonable to ask *which hypotheses are invariant* under choice of ULI constraints (or equivalently, under a choice of the scale of the latent variable), and *which are not*.
- Unless a particular choice of constraint (or latent variable variance) has a specific substantive meaning, a hypothesis that is not invariant under a choice of constraints will be difficult if not impossible to interpret.
- For example, is the hypothesis that $\lambda_{1,1}$ equals $\lambda_{2,1}$ in the the general model invariant under a change of scale of the latent variables? (C.P.)

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- From the preceding analysis, it would seem that the answer is yes, since any change in the ULI constraint would be reflected proportionally in coefficients $\lambda_{1,1}$ and $\lambda_{2,1}$.
- The choice of the particular value employed in the identifying constraint has no effect on this hypothesis.
- Another way of putting it is that the particular value of the variance of η_1 has no effect on the truth or falsity of the hypothesis.

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- Similarly, the hypothesis that $\lambda_{3,2}$ and $\lambda_{4,2}$ are equal is invariant under choice of the fixed value employed in an identifying constraint on the variance of η_2.

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- We have established there are hypotheses about the model coefficients that are invariant under the choice of value we fix latent variable variances to, so long as the constraints are only to achieve identification.
- It seems reasonable to suggest that, if a hypothesis is invariant under the choice of the fixed value used in the identifying constraint, then the hypothesis might be considered meaningful when the value of 1.0 typically used in the ULI is used.

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- Consider the hypothesis

$$H_0 : \lambda_{2,1} = \lambda_{4,2}$$

Is the truth status of this hypothesis invariant under a choice of identification constraint?

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- The answer is "no." This hypothesis is not invariant under the choice of fixed value employed in the identifying constraint on $\lambda_{1,1}$.
- Doubling the fixed value of $\lambda_{1,1}$ doubles the value of $\lambda_{2,1}$ while leaving $\lambda_{4,2}$ unchanged. In this case, the hypothesis is not *invariant under change of scale of the latent variables*.
- This has an important implication: When ULI constraints are used, the variances of factors are fixed at a value that is essentially an accident. The factors will almost never have a variance of 1.
- Consequently, the model that includes the restriction $\lambda_{2,1} = \lambda_{4,2}$ *will not be the same model* when unit variances are imposed as when ULI constraints are imposed.

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- On the next slide, we have a table from Steiger(2002) listing the Σ-constraints for the two models.
- You can see that the two models have identical constraints except for one.
- Note that the model using ULI constraints has a more restrictive final Σ-constraint than the model using unit variances. Some covariance matrices that satisfy the constraint on the far right will not satisfy the constraint that $\sigma_{4,1} = \sigma_{3,2}$.

ULI Constraints and Identification

Invariance of Hypotheses under Choice of Constraints

- So we see that some hypotheses might make sense when ULI constraints (or other arbitrary identification constraints) are employed, while others might not make sense.
- The impression given by many textbooks is that ULI constraints are automatic and, in a sense, arbitrary. They might be within the simple context of a few textbook examples, but in the larger framework of structural equation modeling in full generality, they might not be.

ULI Constraints and Identification

Some Questions to Ask

- In analyzing whether a ULI constraint (or set of constraints) is truly arbitrary, we should ask questions like these:
  1. Does the goodness-of-fit statistic remain invariant under the choice of fixed value employed in the identifying constraint? That is, if we change the 1.0 to some other number, does the value remain constant?
  2. Does the goodness-of-fit statistic remain invariant under the choice of which manifest variable is the reference variable?
  3. Do the relative sizes of path coefficients leading to the latent variable remain invariant under the choice of the fixed value employed in the identifying constraint?
  4. Do the relative sizes of path coefficients leading from the latent variable remain invariant under the choice of the reference variable?

When Constraints Interact

Problems with the Chi-Square Difference Test

- In structural equation modeling, a hypothesis of equality of path coefficients is usually tested with a chi-square difference test.
- Unfortunately, some chi-square difference tests do not perform in the intended manner—they are compromised by a phenomenon I call constraint interaction.

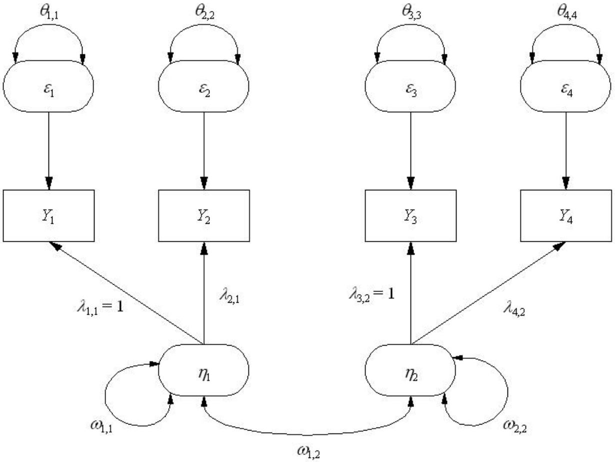
When Constraints Interact

Problems with the Chi-Square Difference Test

- To examine constraint interaction in a simple context, suppose one wished to test the hypothesis that $\lambda_{2,1}$ and $\lambda_{4,2}$ are equal (perhaps in an effort to convince people that they are *not* equal) in the simple factor model on the next slide.

When Constraints Interact

Problems with the Chi-Square Difference Test



When Constraints Interact

Problems with the Chi-Square Difference Test

- The difference test would normally proceed as follows:
  1. First fit a version of this model with the two loadings constrained to be equal.
  2. Then fit the model without the single equality constraint.
  3. The difference between the two χ^2 statistics is a χ^2 with 1 degree of freedom.
- However, what seems to be straightforward turns out not to be.

When Constraints Interact

A Challenging Example

- Consider the following matrix:

$$\boldsymbol{\Sigma} = \begin{bmatrix} 3.490 & 0.420 & 0.112 & 0.168 \\ 0.420 & 3.360 & 0.096 & 0.144 \\ 0.112 & 0.096 & 1.160 & 0.240 \\ 0.168 & 0.144 & 0.240 & 1.360 \end{bmatrix} \tag{12}$$

- As you could quickly verify by loading MPLUS or LISREL, this matrix perfectly fits the factor model where, instead of ULI constraints, the factors are constrained to have unit variances and $\lambda_{2,1} = \lambda_{4,2}$. The model equations are shown in the following slide.

When Constraints Interact

A Challenging Example

- The model can be written as:

$$\mathbf{\Lambda_y} = \begin{bmatrix} \lambda_{1,1} & 0 \\ \lambda_{2,1} & 0 \\ 0 & \lambda_{3,2} \\ 0 & \lambda_{2,1} \end{bmatrix} \tag{13}$$

$$\mathbf{\Omega} = \begin{bmatrix} 1 & \omega_{2,1} \\ \omega_{2,1} & 1 \end{bmatrix} \tag{14}$$

$$\mathbf{\Theta}_{\epsilon} = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix} \tag{15}$$

When Constraints Interact

A Challenging Example

- We can verify that the following numerical values perfectly reproduce the $\mathbf{\Sigma}$ of Equation 12 via the confirmatory factor analysis formula $\mathbf{\Sigma} = \mathbf{\Lambda_y \Omega \Lambda_y'} + \mathbf{\Theta}_\epsilon$.

$$\mathbf{\Lambda_y} = \begin{bmatrix} .7 & 0 \\ .6 & 0 \\ 0 & .4 \\ 0 & .6 \end{bmatrix} \tag{16}$$

$$\mathbf{\Omega} = \begin{bmatrix} 1 & .4 \\ .4 & 1 \end{bmatrix} \tag{17}$$

$$\mathbf{\Theta}_\epsilon = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

When Constraints Interact

A Challenging Example

- Here are the calculations in R

```
> Lambda.y <- matrix(c(.7,.6,0,0,0,0,.4,.6),4,2)
> Omega <-  matrix(c(1,.4,.4,1),2,2)
> Theta.epsilon <- diag(c(3,3,1,1))
> Lambda.y
     [,1] [,2]
[1,]  0.7  0.0
[2,]  0.6  0.0
[3,]  0.0  0.4
[4,]  0.0  0.6
> Omega
     [,1] [,2]
[1,]  1.0  0.4
[2,]  0.4  1.0
> Theta.epsilon
     [,1] [,2] [,3] [,4]
[1,]    3    0    0    0
[2,]    0    3    0    0
[3,]    0    0    1    0
[4,]    0    0    0    1
> Sigma <- Lambda.y %*% Omega %*% t(Lambda.y) + Theta.epsilon
> Sigma
      [,1]  [,2]  [,3]  [,4]
[1,] 3.490 0.420 0.112 0.168
[2,] 0.420 3.360 0.096 0.144
[3,] 0.112 0.096 1.160 0.240
[4,] 0.168 0.144 0.240 1.360
```

When Constraints Interact

A Challenging Example

- So we see that a CFA model with two loadings constrained to be equal and the factors constrained to have unit variances fits perfectly.
- However, an alternative way of parameterizing such a model might *seem* to be to use two ULI constraints, as in the equations on the next slide.

When Constraints Interact

A Challenging Example

$$\boldsymbol{\Lambda}_{\mathbf{y}} = \begin{bmatrix} 1 & 0 \\ \lambda_{2,1} & 0 \\ 0 & 1 \\ 0 & \lambda_{2,1} \end{bmatrix} \tag{19}$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_{1,1} & \omega_{2,1} \\ \omega_{2,1} & \omega_{2,2} \end{bmatrix} \tag{20}$$

$$\boldsymbol{\Theta}_{\epsilon} = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix} \tag{21}$$

When Constraints Interact

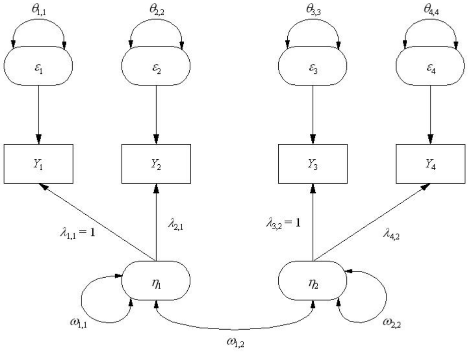
A Challenging Example

- Surprise!
- When you fit the model using ULI constraints and the equality constraint with Mplus, the model does not fit the $\boldsymbol{\Sigma}$ of Equation 12 perfectly!
- It is not the same model.

When Constraints Interact

A Challenging Example

- Looking carefully at the model diagram, we can see that, when implemented with ULI constraints, it not only forces $\lambda_{2,1}$ and $\lambda_{4,2}$ to be equal, it also requires that $\lambda_{1,1}$ and $\lambda_{3,2}$ also be equal, since they are both constrained to be equal to 1.
- However, $\mathbf{\Sigma}$ was not generated by a $\mathbf{\Lambda}_y$ with $\lambda_{1,1}$ and $\lambda_{3,2}$ equal (their values were .70 and .40), and so the model fails to fit perfectly.



When Constraints Interact

A Challenging Example

- We have seen that two seemingly equivalent methods for identifying factor variances are not always equivalent.
- Without the equality constraint $\lambda_{2,1} = \lambda_{4,2}$ the two methods for fixing variances *are* equivalent.
- When this equal loadings constraint is added, the two methods yield models that are not $\boldsymbol{\Sigma}$-equivalent.
- What this means, in turn, is that a χ^2 difference test of the hypothesis that $\lambda_{2,1} = \lambda_{4,2}$ will produce different results, depending on whether the model is parameterized with ULI constraints or with standardized (unit variance) latent variables.

When Constraints Interact

A Challenging Example

- On the next slides, we will examine Mplus output for several conditions, with and without the use of ULI constraints, and with and without the restriction that $\lambda_{2,1} = \lambda_{4,2}$.

When Constraints Interact

A Challenging Example

```
INPUT INSTRUCTIONS

  TITLE: Simple Confirmatory Steiger(2002)
  DATA: FILE is 4x2.dat;
        TYPE IS COVARIANCE;
        NOBSERVATIONS=101;
  VARIABLE: NAMES ARE Y1-Y4;
  MODEL: ETA1 BY Y1 Y2;
         ETA2 BY Y3 Y4;
         ETA1 WITH ETA2;

Chi-Square Test of Model Fit

          Value                              0.000
          Degrees of Freedom                     1
          P-Value                           1.0000

MODEL RESULTS

                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 ETA1     BY
    Y1                 1.000      0.000    999.000    999.000
    Y2                 0.857      1.386      0.618      0.536

 ETA2     BY
    Y3                 1.000      0.000    999.000    999.000
    Y4                 1.500      2.386      0.629      0.530

 ETA1     WITH
    ETA2               0.111      0.176      0.631      0.528

 Variances
    ETA1               0.485      0.873      0.556      0.578
    ETA2               0.158      0.269      0.589      0.556

 Residual Variances
    Y1                 2.970      0.936      3.175      0.001
    Y2                 2.970      0.744      3.994      0.000
    Y3                 0.990      0.292      3.396      0.001
    Y4                 0.990      0.593      1.670      0.095
```

When Constraints Interact

A Challenging Example

```
INPUT INSTRUCTIONS

  TITLE: Simple Confirmatory Steiger(2002)
  DATA: FILE is 4x2N10.dat;
        TYPE IS FULLCOV;
        NOBSERVATIONS=10;
  VARIABLE: NAMES ARE Y1-Y4;
  MODEL: ETA1 BY Y1 Y2;
         ETA2 BY Y3 Y4;
         ETA1 WITH ETA2;

Chi-Square Test of Model Fit

          Value                            0.000
          Degrees of Freedom                   1
          P-Value                         1.0000

MODEL RESULTS

                                                  Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 ETA1     BY
    Y1                 1.000      0.000    999.000    999.000
    Y2                 0.857      4.404      0.195      0.846

 ETA2     BY
    Y3                 1.000      0.000    999.000    999.000
    Y4                 1.500      7.582      0.198      0.843

 ETA1     WITH
    ETA2               0.101      0.508      0.199      0.843

 Variances
    ETA1               0.441      2.523      0.175      0.861
    ETA2               0.144      0.777      0.185      0.853

 Residual Variances
    Y1                 2.700      2.703      0.999      0.318
    Y2                 2.700      2.148      1.257      0.209
    Y3                 0.900      0.842      1.069      0.285
    Y4                 0.900      1.712      0.526      0.599
```

When Constraints Interact

A Challenging Example

- Most structural equation modeling programs will return the same model estimates for a given input covariance matrix, regardless of the sample size.
- Mplus, on the other hand, gives estimates that can vary as a function of sample size.
- Why is this so?

When Constraints Interact

A Challenging Example

```
> S <- matrix(c(3.490, 0.420, 0.112, 0.168,
+ 0.420, 3.360, 0.096, 0.144,
+ 0.112, 0.096, 1.160, 0.240,
+ 0.168, 0.144, 0.240, 1.360),4,4,byrow=TRUE)
> fixS <- function(S,n){
+   return(S*n/(n-1))
+ }
> fixS(S,10)
```

```
          [,1]      [,2]      [,3]      [,4]
[1,] 3.8777778 0.4666667 0.1244444 0.1866667
[2,] 0.4666667 3.7333333 0.1066667 0.1600000
[3,] 0.1244444 0.1066667 1.2888889 0.2666667
[4,] 0.1866667 0.1600000 0.2666667 1.5111111
```

When Constraints Interact

A Challenging Example

```
INPUT INSTRUCTIONS

  TITLE: Simple Confirmatory Steiger(2002)
  DATA: FILE is 4x2N10.dat;
        TYPE IS FULLCOV;
        NOBSERVATIONS=10;
  VARIABLE: NAMES ARE Y1-Y4;
  MODEL: ETA1 BY Y1 Y2;
         ETA2 BY Y3 Y4;
         ETA1 WITH ETA2;

Chi-Square Test of Model Fit

          Value                          0.000
          Degrees of Freedom                 1
          P-Value                       1.0000

MODEL RESULTS

                                                 Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 ETA1     BY
    Y1                 1.000      0.000    999.000    999.000
    Y2                 0.857      4.405      0.195      0.846

 ETA2     BY
    Y3                 1.000      0.000    999.000    999.000
    Y4                 1.500      7.580      0.198      0.843

 ETA1     WITH
    ETA2               0.112      0.564      0.199      0.843

 Variances
    ETA1               0.490      2.802      0.175      0.861
    ETA2               0.160      0.863      0.185      0.853

 Residual Variances
    Y1                 3.000      3.003      0.999      0.318
    Y2                 3.000      2.387      1.257      0.209
    Y3                 1.000      0.936      1.069      0.285
    Y4                 1.000      1.902      0.526      0.599
```

When Constraints Interact

A Challenging Example

```
INPUT INSTRUCTIONS

  TITLE: 4x2 Equal Loadings and ULI Steiger(2002)
  DATA: FILE is 4x2.dat;
        TYPE IS COVARIANCE;
        NOBSERVATIONS=101;
  VARIABLE: NAMES ARE Y1-Y4;
  MODEL: ETA1 BY Y1;
         ETA1 BY Y2(1);
         ETA2 BY Y3;
         ETA2 BY Y4(1);
         ETA1 WITH ETA2;
  OUTPUT: STANDARDIZED;

Chi-Square Test of Model Fit

          Value                           0.062
          Degrees of Freedom                  2
          P-Value                        0.9694
MODEL RESULTS

                                                 Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 ETA1     BY
    Y1                 1.000      0.000    999.000    999.000
    Y2                 1.154      1.303      0.886      0.376

 ETA2     BY
    Y3                 1.000      0.000    999.000    999.000
    Y4                 1.154      1.303      0.886      0.376

 ETA1     WITH
    ETA2               0.109      0.173      0.632      0.527

 Variances
    ETA1               0.359      0.504      0.713      0.476
    ETA2               0.206      0.258      0.800      0.424

 Residual Variances
    Y1                 3.089      0.648      4.767      0.000
    Y2                 2.854      0.756      3.777      0.000
    Y3                 0.944      0.281      3.362      0.001
    Y4                 1.070      0.362      2.952      0.003
```

When Constraints Interact

A Challenging Example

```
INPUT INSTRUCTIONS

  TITLE: 4x2 Equal Loadings and Unit Variances Steiger(2002)
  DATA: FILE is 4x2.dat;
        TYPE IS COVARIANCE;
        NOBSERVATIONS=101;
  VARIABLE: NAMES ARE Y1-Y4;
  MODEL: ETA1 BY Y1* ;
         ETA1 BY Y2*(1);
         ETA2 BY Y3*;
         ETA2 BY Y4*(1);
         ETA1 WITH ETA2;
         ETA1@1;
         ETA2@1;
  OUTPUT: STANDARDIZED;

Chi-Square Test of Model Fit

          Value                              0.000
          Degrees of Freedom                     2
          P-Value                           1.0000

MODEL RESULTS

                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 ETA1     BY
    Y1                 0.697      0.563      1.238      0.216
    Y2                 0.597      0.366      1.630      0.103

 ETA2     BY
    Y3                 0.398      0.284      1.399      0.162
    Y4                 0.597      0.366      1.630      0.103

 ETA1     WITH
    ETA2               0.400      0.396      1.009      0.313

 Variances
    ETA1               1.000      0.000    999.000    999.000
    ETA2               1.000      0.000    999.000    999.000

 Residual Variances
    Y1                 2.970      0.852      3.486      0.000
    Y2                 2.970      0.606      4.901      0.000
    Y3                 0.990      0.252      3.923      0.000
    Y4                 0.990      0.457      2.168      0.030
```

When Constraints Interact

An Unnecessary Constraint

- Before discussing this phenomenon in more detail, we should digress briefly to note an important, and apparently unnoticed fact that helps explain the source of the nonequivalency.
- When the model includes the constraint that $\lambda_{2,1} = \lambda_{4,2}$, only one ULI constraint (or, alternatively, unit variance constraint) is necessary to establish identification. For example, consider again the model of Equation 19.

When Constraints Interact

An Unnecessary Constraint

- Although the frequently cited “rule” for identifying variables might lead one to believe that two ULI constraints are necessary to identify this model, it actually will remain identified if one of the ULI constraints is relaxed.
- Suppose we relax the constraint on $\lambda_{1,1}$, and allow it to be a free parameter.

When Constraints Interact

An Unnecessary Constraint

- The model equations become

$$\mathbf{\Lambda_y} = \begin{bmatrix} \lambda_{1,1} & 0 \\ \lambda_{2,1} & 0 \\ 0 & 1 \\ 0 & \lambda_{2,1} \end{bmatrix} \tag{22}$$

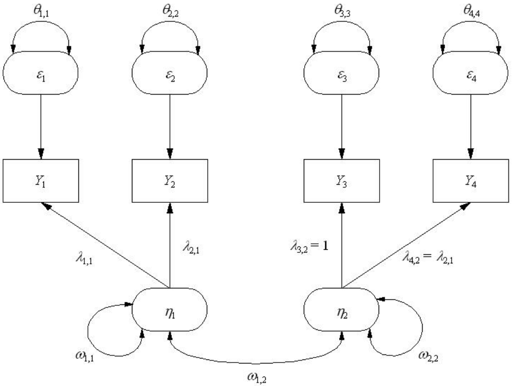
$$\mathbf{\Omega} = \begin{bmatrix} \omega_{1,1} & \omega_{2,1} \\ \omega_{2,1} & \omega_{2,2} \end{bmatrix} \tag{23}$$

$$\mathbf{\Theta}_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix} \tag{24}$$

When Constraints Interact

An Unnecessary Constraint

- Let's look at the path diagram and use the pipeline metaphor to see why the variances of the common factors are still identified despite the removal of one of the ULI constraints. Who can explain it? (C.P.)



When Constraints Interact

An Unnecessary Constraint

- Imagine that the variances of η_1 and η_2 are identified at some value, and that the model fits perfectly.
- Now ask the question, can we vary the variance of either η_1 or η_2 and compensate for it by adjusting other model coefficients?
- First imagine that we double the standard deviation of η_1. We could try to compensate for this by halving all coefficients attached to η_1 (i.e., $\lambda_{1,1}$, $\lambda_{2,1}$, and $\omega_{1,2}$).
- Note however that halving the value of $\lambda_{2,1}$ would require halving the value of $\lambda_{4,2}$ because of the equality constraint. However, this cannot be done without changing the fit of the model.

When Constraints Interact

An Unnecessary Constraint

- We have discovered a surprising fact. The equality constraint on the λ's not only constrains them to be equal, it also fixes the variance of η_1 to a particular value. What value? The value is essentially arbitrary, i.e., it might be described as "whatever value occurs when $\lambda_{3,2}$ is fixed at 1, and $\lambda_{2,1}$ and $\lambda_{4,2}$ are constrained to be the same free parameter."

When Constraints Interact

An Unnecessary Constraint

- Consequently, once the equality constraint on the λ's is in place, the unnecessary second ULI (or unit variance) constraint actually overconstrains the model beyond what is necessary for identification.
- The effect of the unnecessary additional constraint depends on its type — adding the second ULI constraint forces parallel λ's to be equal, while adding a second unit variance constraint forces the factor variances to be standardized.

When Constraints Interact

An Unnecessary Constraint

```
INPUT INSTRUCTIONS

  TITLE: 4x2 Equal Loadings and ULI Steiger(2002)
  DATA: FILE is 4x2.dat;
        TYPE IS COVARIANCE;
        NOBSERVATIONS=101;
  VARIABLE: NAMES ARE Y1-Y4;
  MODEL: ETA1 BY Y1;
         ETA1 BY Y2(1);
         ETA2 BY Y3*;
         ETA2 BY Y4(1);
         ETA1 WITH ETA2;
  OUTPUT: STANDARDIZED;

Chi-Square Test of Model Fit

          Value                              0.000
          Degrees of Freedom                     1
          P-Value                           0.9993

STD Standardization

                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 ETA1     BY
    Y1                 0.696      0.626      1.111      0.266
    Y2                 0.597      0.545      1.097      0.273

 ETA2     BY
    Y3                 0.398      0.338      1.178      0.239
    Y4                 0.597      0.494      1.208      0.227

 ETA1     WITH
    ETA2               0.400      0.404      0.990      0.322

 Variances
    ETA1               1.000      0.000    999.000    999.000
    ETA2               1.000      0.000    999.000    999.000

 Residual Variances
    Y1                 2.971      0.935      3.179      0.001
    Y2                 2.970      0.744      3.992      0.000
    Y3                 0.990      0.291      3.397      0.001
    Y4                 0.990      0.593      1.670      0.095
```

When Constraints Interact

A Damaging Side-Effect

- Unfortunately, the χ^2 difference test for equal λ's cannot be performed unless the identification constraints are kept constant for the two tests, because if the unnecessary identification constraint is removed, the two models will have the same degrees of freedom, and will not be nested.

When Constraints Interact

Some Implications

- The results discussed in this section have several important implications:
  1. A χ^2 difference test for equal factor loadings on different factors is not "scale-free," i.e., it depends upon the scaling of the factors involved.
  2. If loadings on different factors are constrained to be equal, then the factor variances may be identified without a ULI constraint being employed on every factor.
  3. Conditions (1) and (2) may generalize to many situations other than the simple one discussed here. They will certainly generalize to any situation that can be conceptualized as a factor model.
  4. When the χ^2 difference test is not scale-invariant, choice of a particular scale might be based on substantive grounds. If no reasonable substantive grounds exist, then such a test may not be meaningful.

When Constraints Interact

Some Implications

- The lessons learned in the context of a simple confirmatory factor analysis model generalize to more complex structural equation models. In my 2002 article, "When Constraints Interact," I demonstrate how some classic models, including a well-known example from the early LISREL manuals, falls prey to this problem.

A Simple Numerical Approach To Detecting Constraint Interaction

- Constraint interaction, when it occurs, has an important implication — equality of two coefficients will not be invariant under changes of scale of the latent variables.
- Whether a hypothesis test "makes sense" under such circumstances is an issue to be addressed in a subsequent section.
- Here, we concentrate on a simple approach to detecting constraint interaction in practice.

A Simple Numerical Approach To Detecting Constraint Interaction

- Recall that, when constraints interact, a model that fits perfectly under one parameterization will not fit perfectly under another.
- This is because the model itself is not invariant under changes of scale of the latent variables.
- One way of testing whether a model is sensitive to the scale of the latent variables is to test whether the fit of the model is sensitive to the value used in the ULI constraint.
- Generally, of course, this value is set equal to 1.
- For models that are invariant under changes of scale of the latent variables, the magnitude of this value may be varied to any nonzero number without affecting model fit. However, for models that are not invariant under changes of scale, varying the magnitude of the value will affect model fit.

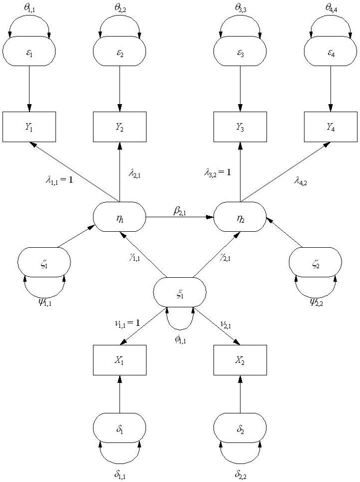
A Simple Numerical Approach To Detecting Constraint Interaction

- Consequently, a simple way of detecting constraint interaction is the following:
  1. For the model with equality constraints, compute a χ^2 fit statistic with the standard ULI constraints in place;
  2. Then, alter the model so that one of these constraints is altered, say, to a value of 2 instead of 1;
  3. Recompute the χ^2 statistic;
  4. If the two χ^2 statistics are not identical within rounding error, and convergence has occurred in each case, then the model is not invariant under changes of scale of the latent variables.

Investigating Constraint Interaction in the General Model

- One of the lessons learned in the preceding discussion is that equality constraints that hold for one scaling of a set of variables need not hold for another scaling.
- One "natural" scaling that is sometimes considered is a *completely standardized* model, in which all variables, both manifest and latent, have unit variances.
- Consider again the general model shown on next slide.
- There were several barriers to obtaining a correct completely standardized solution with correct standard errors.
  1. Most programs couldn't analyze correlations successfully.
  2. Although it was straightforward to fix exogenous factor variances at 1 (and remove the ULI constraint), it was not possible to control the variance of the endogenous latent variables. Their variances were simply a consequence of other model coefficients, variances, and covariances.
- The typical way most structural equation modeling programs achieved a fully standardized solution was as follows:
  1. Fit the model using ULI constraints.
  2. Use standard regression algebra to convert the model to standarized form.
  3. Do not report standard errors.

Investigating Constraint Interaction in the General Model



Investigating Constraint Interaction in the General Model

- Unfortunately, this approach can fail with certain models and data sets.
- Suppose that we wish to fit a completely standardized version of the model on the preceding slide, while incorporating the restriction that $\gamma_{1,1} = \gamma_{2,1}$.
- If the standard approach is taken (ULI constraints on both factors, followed by post-hoc computation of the model coefficients), the model will not be assessed properly, because two ULI constraints are not necessary to identify the variances of the endogenous latent variables.
- Let's examine how this plays out in a numerical example from Steiger(2002).

Investigating Constraint Interaction in the General Model

- Steiger(2002, Equation 13) gave the covariance matrix shown below as input to the General Model.
- He asserted that a completely standardized model fit this matrix with all ULI constraints replaced by unit variance constraints on the factors, *and* with $\gamma_{1,1} = \gamma_{2,1}$.

$$\Sigma = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 & X_1 & X_2 \\ 0.400000 & & & & & \\ 0.090000 & 0.400000 & & & & \\ 0.050000 & 0.060000 & 0.400000 & & & \\ 0.060000 & 0.072000 & 0.124026 & 0.400000 & & \\ 0.058333 & 0.070000 & 0.077778 & 0.093333 & 0.400000 & \\ 0.050000 & 0.060000 & 0.066667 & 0.080000 & 0.090000 & 0.400000 \end{bmatrix}.$$

Investigating Constraint Interaction in the General Model

- The SEPATH prgram uses a constrained estimation procedure that automatically iterates to a solution with unit variances, automatically analyzes correlations correctly, and, in this case of artificial data, finds perfect fit – a χ^2 value of 0.
- Here are the results.

	Model Estimates (GeneralModelTestSetV			
	Parameter Estimate	**Standard Error**	T Statistic	Prob. Level
(XI1)-1->[X1]	0.512	0.046	11.030	0.000
(XI1)-2->[X2]	0.439	0.044	10.007	0.000
(XI1)-{1}-(XI1)				
(DELTA1)-->[X1]				
(DELTA2)-->[X2]				
(DELTA1)-3-(DELTA1)	0.737	0.048	15.495	0.000
(DELTA2)-4-(DELTA2)	0.807	0.039	20.941	0.000
(ETA1)-98->[Y1]	0.433	0.049	8.848	0.000
(ETA1)-5->[Y2]	0.520	0.053	9.782	0.000
(ETA2)-99->[Y3]	0.508	0.042	12.002	0.000
(ETA2)-6->[Y4]	0.610	0.046	13.231	0.000
(EPSILON1)-->[Y1]				
(EPSILON2)-->[Y2]				
(EPSILON3)-->[Y3]				
(EPSILON4)-->[Y4]				
(EPSILON1)-7-(EPSILON1)	0.813	0.042	19.171	0.000
(EPSILON2)-8-(EPSILON2)	0.730	0.055	13.224	0.000
(EPSILON3)-9-(EPSILON3)	0.742	0.043	17.223	0.000
(EPSILON4)-10-(EPSILON4)	0.628	0.056	11.164	0.000
(ZETA1)-->(ETA1)				
(ZETA2)-->(ETA2)				
(ZETA1)-11-(ZETA1)	0.568	0.117	4.873	0.000
(ZETA2)-12-(ZETA2)	0.432	0.081	5.352	0.000
(XI1)-13->(ETA1)	0.657	0.089	7.416	0.000
(XI1)-13->(ETA2)	0.657	0.089	7.416	0.000
(ETA1)-15->(ETA2)	0.136	0.120	1.132	0.258

Investigating Constraint Interaction in the General Model

- If we do a standard setup in MPLUS, using ULI constraints on all variables, and calling for a standardized solution, with $\gamma_{1,1} = \gamma_{1,2}$, we get the following.
- The perfect fit is not found.

```
INPUT INSTRUCTIONS

  TITLE: TEST DATA FROM STEIGER(2002)
  DATA: FILE IS Steiger02Test.dat;
  TYPE IS COVARIANCE;
  NOBSERVATIONS = 932;
  VARIABLE: NAMES ARE X1 X2 Y1 Y2 Y3 Y4;
  MODEL: XI1 BY X1 X2;
  ETA1 BY Y1 Y2;
  ETA2 BY Y3 Y4;
  ETA1 ON XI1 (1);
  ETA2 ON XI1 (1);
  ETA2 ON ETA1;
  OUTPUT: STANDARDIZED;

Chi-Square Test of Model Fit

          Value                              1.924
          Degrees of Freedom                     7
          P-Value                           0.9639
```

- As we see on the next slide, the estimates are also wrong. In particular, the loadings of ETA1 and ETA2 on XI1 are not the same, and neither is equal to the correct value of 0.657.

Investigating Constraint Interaction in the General Model

```
STDYX Standardization

                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 XI1      BY
    X1                 0.430      0.049      8.827      0.000
    X2                 0.512      0.054      9.484      0.000

 ETA1     BY
    Y1                 0.489      0.041     11.796      0.000
    Y2                 0.633      0.047     13.358      0.000

 ETA2     BY
    Y3                 0.523      0.048     10.971      0.000
    Y4                 0.434      0.044      9.934      0.000

 ETA1     ON
    XI1                0.520      0.069      7.493      0.000

 ETA2     ON
    XI1                0.483      0.080      6.024      0.000
    ETA1               0.451      0.094      4.819      0.000

 Variances
    XI1                1.000      0.000    999.000    999.000

 Residual Variances
    X1                 0.815      0.042     19.440      0.000
    X2                 0.738      0.055     13.360      0.000
    Y1                 0.761      0.041     18.742      0.000
    Y2                 0.599      0.060      9.989      0.000
    Y3                 0.727      0.050     14.576      0.000
    Y4                 0.812      0.038     21.415      0.000
    ETA1               0.729      0.072     10.082      0.000
    ETA2               0.336      0.116      2.901      0.004
```

Investigating Constraint Interaction in the General Model

- If we remove one of the ULI constraints and re-run, we get almost the same results as SEPATH. Because fit is perfect, MPLUS recovers the correct estimates and standard errors, except for one anomaly.

Investigating Constraint Interaction in the General Model

```
INPUT INSTRUCTIONS

  TITLE: TEST DATA FROM STEIGER(2002)
  DATA: FILE IS Steiger02Test.dat;
  TYPE IS COVARIANCE;
  NOBSERVATIONS = 931;
  VARIABLE: NAMES ARE Y1 Y2 Y3 Y4 X1 X2;
  MODEL:
  XI1 BY X1*;
  XI1 BY X2*;
  XI1@1;
  ETA1 BY Y1*;
  ETA1 BY Y2*;
  ETA2 BY Y3@1;
  ETA2 BY Y4*;
  ETA1 ON XI1 (1);
  ETA2 ON XI1 (1);
  ETA2 ON ETA1;
  OUTPUT: STANDARDIZED;

Chi-Square Test of Model Fit

          Value                              0.000
          Degrees of Freedom                     6
          P-Value                           1.0000
```

Investigating Constraint Interaction in the General Model

```
STANDARDIZED MODEL RESULTS


STDYX Standardization

                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 XI1      BY
    X1                 0.512      0.048     10.631      0.000
    X2                 0.439      0.044      9.877      0.000

 ETA1     BY
    Y1                 0.433      0.049      8.812      0.000
    Y2                 0.520      0.055      9.442      0.000

 ETA2     BY
    Y3                 0.508      0.042     11.998      0.000
    Y4                 0.610      0.046     13.133      0.000

 ETA1     ON
    XI1                0.657      0.091      7.209      0.000

 ETA2     ON
    XI1                0.657      0.160      4.107      0.000
    ETA1               0.136      0.165      0.824      0.410

 Variances
    XI1                1.000      0.000    999.000    999.000

 Residual Variances
    Y1                 0.812      0.043     19.092      0.000
    Y2                 0.730      0.057     12.766      0.000
    Y3                 0.742      0.043     17.220      0.000
    Y4                 0.628      0.057     11.082      0.000
    X1                 0.738      0.049     14.936      0.000
    X2                 0.807      0.039     20.668      0.000
    ETA1               0.568      0.120      4.737      0.000
    ETA2               0.432      0.105      4.126      0.000
```

Investigating Constraint Interaction in the General Model

- Notice that, although the estimates for the two gammas are equal, the standard errors are not.
- The constrained estimation procedure has a standard error of 0.089 on each coefficient, because with the constrained estimation procedure, the estimates are always the same (when they are specified to be the same).
- On the other hand, the method with one ULI constraint will generally not find the same solution if fit is not perfect.
- For example the raw test data in `TestDataNonPerfectFit.txt` generates a standardized solution shown on the next slide with a χ^2 of 10.33 with constrained estimation, while the same data produce a standardized solution in Mplus that does not have equal gammas after transformation to standardized form. The χ^2 of 7.56 obtained by Mplus is the same as the value obtained by SEPATH using the unconstrained estimation approach.

Investigating Constraint Interaction in the General Model

	Model Estimates (Spreadsheet11)			
	Parameter Estimate	Standard Error	T Statistic	Prob. Level
(XI1)-1->[X1]	0.572	0.045	12.614	0.000
(XI1)-2->[X2]	0.408	0.041	9.890	0.000
(XI1)-{1}-(XI1)				
(DELTA1)-->[X1]				
(DELTA2)-->[X2]				
(DELTA1)-3-(DELTA1)	0.673	0.052	12.986	0.000
(DELTA2)-4-(DELTA2)	0.833	0.034	24.695	0.000
(ETA1)-98->[Y1]	0.410	0.047	8.781	0.000
(ETA1)-5->[Y2]	0.538	0.053	10.142	0.000
(ETA2)-99->[Y3]	0.490	0.041	11.848	0.000
(ETA2)-6->[Y4]	0.617	0.046	13.442	0.000
(EPSILON1)-->[Y1]				
(EPSILON2)-->[Y2]				
(EPSILON3)-->[Y3]				
(EPSILON4)-->[Y4]				
(EPSILON1)-7-(EPSILON1)	0.832	0.038	21.721	0.000
(EPSILON2)-8-(EPSILON2)	0.710	0.057	12.435	0.000
(EPSILON3)-9-(EPSILON3)	0.759	0.041	18.704	0.000
(EPSILON4)-10-(EPSILON4)	0.619	0.057	10.931	0.000
(ZETA1)-->(ETA1)				
(ZETA2)-->(ETA2)				
(ZETA1)-11-(ZETA1)	0.506	0.126	4.020	0.000
(ZETA2)-12-(ZETA2)	0.418	0.079	5.290	0.000
(XI1)-13->(ETA1)	0.703	0.090	7.845	0.000
(XI1)-13->(ETA2)	0.703	0.090	7.845	0.000
(ETA1)-15->(ETA2)	0.083	0.120	0.688	0.491

Investigating Constraint Interaction in the General Model

```
INPUT INSTRUCTIONS

  TITLE: TEST DATA FROM STEIGER(2002)
  DATA: FILE IS TestDataNonPerfectFit.txt;
  TYPE IS INDIVIDUAL;
  VARIABLE: NAMES ARE Y1 Y2 Y3 Y4 X1 X2;
  MODEL:
  XI1 BY X1*;
  XI1 BY X2*;
  XI1@1;
  ETA1 BY Y1*;
  ETA1 BY Y2*;
  ETA2 BY Y3@1;
  ETA2 BY Y4*;
  ETA1 ON XI1 (1);
  ETA2 ON XI1 (1);
  ETA2 ON ETA1;
  OUTPUT: STANDARDIZED;

Chi-Square Test of Model Fit

          Value                              7.569
          Degrees of Freedom                     6
          P-Value                           0.2714
```

Investigating Constraint Interaction in the General Model

```
STDYX Standardization

                                                    Two-Tailed
                    Estimate       S.E.  Est./S.E.    P-Value

 XI1      BY
    X1                 0.605      0.052     11.694      0.000
    X2                 0.414      0.042      9.795      0.000

 ETA1     BY
    Y1                 0.405      0.046      8.812      0.000
    Y2                 0.514      0.053      9.757      0.000

 ETA2     BY
    Y3                 0.490      0.042     11.687      0.000
    Y4                 0.629      0.048     13.203      0.000

 ETA1     ON
    XI1                0.715      0.088      8.098      0.000

 ETA2     ON
    XI1                0.425      0.172      2.477      0.013
    ETA1               0.340      0.176      1.925      0.054

 Intercepts
    Y1                -0.014      0.033     -0.415      0.678
    Y2                 0.011      0.033      0.330      0.741
    Y3                -0.010      0.033     -0.298      0.766
    Y4                 0.023      0.033      0.709      0.478
    X1                -0.070      0.033     -2.133      0.033
    X2                -0.037      0.033     -1.141      0.254

 Variances
    XI1                1.000      0.000    999.000    999.000

 Residual Variances
    Y1                 0.836      0.037     22.476      0.000
    Y2                 0.736      0.054     13.596      0.000
    Y3                 0.760      0.041     18.476      0.000
    Y4                 0.605      0.060     10.092      0.000
    X1                 0.633      0.063     10.105      0.000
    X2                 0.828      0.035     23.615      0.000
    ETA1               0.489      0.126      3.874      0.000
    ETA2               0.497      0.085      5.852      0.000
```